Ranks of (quadratic twists of) elliptic curves with a rational point of order 3

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Basic Definitions

An elliptic curve $E/\mathbb{Q}$ can be given in **WS form** $E_{A,B} = y^2 = x^3 + Ax + B$.

Given $d \neq 0$ not a square, the **quadratic twist** of E is $E^d : dy^2 = x^3 + Ax + B$.

Define the **height** $H(E_{A,B}) := \max\{4|A^3|, 27B^2\}$.

Mordell-Weil $\Rightarrow E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^r$. The **rank** $r(E)$ of E is r.

Average rank of elliptic curves in a family F:

$$\lim_{X\to\infty} \frac{\sum_{E\in F:H(E)<X} r(E)}{\sum_{E\in F:H(E)<X} 1}$$

Quadratic Twisting - Influence on Arithmetic Properties?

Given an elliptic curve, consider its quadratic twist(s).
(How) does this change the arithmetic properties?

Given a family of elliptic curves, consider the family of their quadratic twists.
(How) does this change the arithmetic properties?

- Is the average rank of all elliptic curves over $\mathbb{Q}$ finite?
  - Bhargava-Shankar, 2015: Yes
  - $\leq 1.05 \leq 1.5$
  - Quadratic twists - same proof
- What is the average rank in a general family?
  - What is the average rank in families of elliptic curves having marked points?

Average Size of the 2-Selmer Group

Cohomologically:

$$\mathrm{Sel}_2(E/\mathbb{Q}) = \{x \in H^1(\mathbb{Q}, E[2]) \mid \mathrm{res}_p(x) \in \mathrm{Im}(K_p) \forall p\}$$

where res_p is the restriction of Galois cohomology from $\mathbb{Q}$ to $\mathbb{Q}_p$ and K_p the Kummer map.

It fits into an exact sequence

$$0 \to E(\mathbb{Q})/2E(\mathbb{Q}) \to \mathrm{Sel}_2(E) \to Ш_E[2] \to 0$$

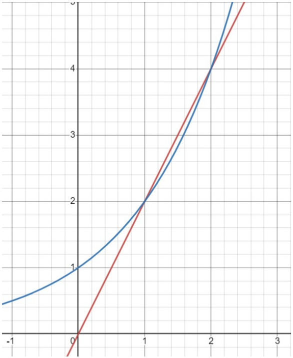
Thus the 2-Selmer rank gives an upper bound for the algebraic rank of E.

From Average Size of Selmer to (a bound on) Average Rank

$$0 \to E(\mathbb{Q})/2E(\mathbb{Q}) \to \mathrm{Sel}_2(E) \to Ш_E[2] \to 0$$

$$\dim_{\mathbb{F}_2}(\mathrm{Sel}_2(E)) = r(E) + \dim_{\mathbb{F}_2}(E(\mathbb{Q})[2]) + \dim_{\mathbb{F}_2}(Ш_E[2])$$

$$2\dim_{\mathbb{F}_2}(\mathrm{Sel}_2(E)) \leq 2^{\dim_{\mathbb{F}_2}(\mathrm{Sel}_2(E))} = |\ \mathrm{Sel}_2(E))\ |$$



Average Size of the 2-Selmer Group - Main Theorem

Consider the family $F = \{y^2 + a_1xy + a_3y = x^3 \mid a_1, a_3 \in \mathbb{Z}, \Delta \neq 0\}$ of elliptic curves over $\mathbb{Q}$ with a marked rational point of order 3.

Theorem (Bhargava-Ho, 2022)

When the elliptic curves in F are ordered by height, $\text{avg}_{E\in F}\ \#\text{Sel}_2(E) \leq 3$.

Corollary: The average rank of elliptic curves in F is finite (≤ 1.5).

New question: Does this result still hold if we consider the family of quadratic twists of elliptic curves in F?

Goal: Show that the average rank is finite and compute an explicit bound.

Parameterisation - Bhargava-Ho

$V = 2 \otimes \mathrm{Sym}_3(2)$ the space of pairs of binary cubic forms, $G = SL_2^2/\mu_2$.

The ring of invariants is generated by a_1, a_3, having degrees 2,6, respectively.

A pair of binary cubic forms $(F_1, F_2) \in V(\mathbb{Q})$ is **locally soluble** if the hyperelliptic curve $z^2 = \mathrm{Disc}_{x,y}(F_1 u + F_2 v)$ is.

Theorem (Bhargava-Ho)

Let $E \in F = \{y^2 + a_1 xy + a_3 y = x^3 \mid a_1, a_3 \in \mathbb{Z}, \Delta \neq 0\}$

There is a bijection

$$\mathrm{Sel}_2(E) \longleftrightarrow G(\mathbb{Q}) \setminus V(\mathbb{Q})^{\text{locally soluble}}_{\mathbf{a_1},\mathbf{a_3}}$$

Counting Outline - Bhargava-Ho

- The computation of $\text{avg}_{E \in F} \#\text{Sel}_2(E)$ is reduced to counting $G(\mathbb{Q})$-orbits that are locally soluble and have bounded integral invariants.
- Select representative integral orbits.
  Need to study $G(\mathbb{Q})$-equivalence classes on $V(\mathbb{Z})$.
- Count of the total number of integral orbits.
- Perform a uniformity estimate and a sieve (for F - Bh-Ho managed only upper bound sieve) to restrict to the locally soluble $\mathbb{Q}$-orbits.
- 1+2=3

Generalisation to Quadratic Twists

Parameterisation on an abstract level.

Integrality?

Counting

Thank you!

Let $E/\mathbb{Q}$ be an elliptic curve. **The 2-Selmer group** $Sel_2(E)$ parameterises isomorphism classes of pairs (C, D), where

C is a genus one curve over $\mathbb{Q}$ satisfying $\mathrm{Pic}^0(C) \cong E$ (torsor)

D is a degree 2 divisor on C

locally soluble: $C(\mathbb{Q}_p) \neq \emptyset$ for all primes p and $C(\mathbb{R}) \neq \emptyset$

$Ш_E = ker(H^1(\mathbb{Q}, E) \to \Pi_p H^1(\mathbb{Q}_p, E))$ the Tate-Shafarevich group